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Factorization of spin-dependent Hamiltonians

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Abstract. We generalize the formalism of the factorization method by introducing spin degrees of freedom, in addition to local potential terms. We compare this construction with conventional factorization techniques of non-relativistic Hamiltonians and the formalism of supersymmetric quantum mechanics. The characteristic isospectrality of the Hamiltonians $H_1 = QR$ and $H_2 = RQ$ is maintained in our more general framework. As illustrative examples we consider two exactly solvable models (related to the harmonic oscillator and the Coulomb problems) and some quasi-exactly solvable problems. In all three cases the zero-energy ground state of H_1 is found to be infinitely degenerate. Further possible generalizations of our approach are also outlined.

1. Introduction

Quantum mechanical problems can be formulated using a wide variety of mathematical tools, such as differential operators, matrices, abstract quantities, etc. The energy spectra of quantum mechanical Hamiltonians constructed this way often exhibit similarities with those of other systems and this fact has been attributed to specific relations between the Hamiltonians. The isospectrality of Hamiltonians has been interpreted in terms of factorization techniques [1], intertwining relations [2, 3], or algebraic constructions [4] also including supersymmetry schemes [5]. These approaches are generally not independent: it has been shown that the relatively new field of supersymmetric quantum mechanics (SUSYQM) is basically a reformulation of the factorization method [6] and its relation to some algebraic approaches has also been discussed [7].

Isospectrality of Hamiltonians can be generated by surprisingly simple constructions that do not even refer to the explicit realization of the operators involved. The intertwining relation [2, 3] between Hamiltonians H_1 and H_2

$$H_1 Q = Q H_2 \tag{1}$$

guarantees, for example, that if there exists an eigenstate, ψ_2, of H_2 with eigenvalue $E^{(2)}$, then $Q\psi_2$ will be an eigenstate of H_1 with the same eigenvalue. (Note that if Q has an inverse, then the above relation can be interpreted as a similarity transformation between H_1 and H_2.) However, based only on (1) nothing more can be said about the energy spectra in general. A particular realization of (1) can be obtained by assuming that the two Hamiltonians are factorized as

$$H_1 = QR \qquad H_2 = RQ. \tag{2}$$

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Then assuming that the eigenfunctions of H_i are ψ_i with eigenvalues $E^{(i)}$, where $i = 1, 2$ the relations

$$H_2(R\psi_1) = RQR\psi_1 = R(H_1\psi_1) = E^{(1)}R\psi_1 \tag{3}$$

$$H_1(Q\psi_2) = QRQ\psi_2 = Q(H_2\psi_2) = E^{(2)}Q\psi_2 \tag{4}$$

follow. With the additional requirement $Q = R^\dagger$ the Hermiticity of the Hamiltonians and the non-negativity of their eigenvalues can also be guaranteed. Furthermore, if ψ_1 is a normalizable eigenfunction of H_1 with eigenvalue $E^{(1)}$, the eigenfunction of H_2 belonging to the same eigenvalue can be given as

$$\psi_2 = (E^{(1)})^{-\frac{1}{2}}(R\psi_1). \tag{5}$$

A special situation occurs when $E^{(1)} = 0$, in which case ψ_2 cannot be defined by (5), so the corresponding energy level at $E = 0$ is missing. Of course, the reverse situation (in which the role of H_1 and H_2 is interchanged) can also be formulated by (4).

It is remarkable that these rather general results of the factorization method [1] hold in their abstract form, without specifying the realization of the operators. In most cases the Schrödinger equation is factorized in one dimension, i.e. on the $x = (-\infty, \infty)$ or the $[0, \infty)$ intervals (in case of radial equations) or on a finite interval. Then the Q and R operators are naturally chosen as linear differential operators of the type $\pm\frac{\mathrm{d}}{\mathrm{d}x} + W(x)$. (Note that multiplying Q with a constant phase factor $\exp(\mathrm{i}\phi)$ and R with $\exp(-\mathrm{i}\phi)$ does not affect the construction outlined above.) Combining operators of the type Q and R with matrices gives rise to various SUSYQM constructions [5]. The N supercharges and the supersymmetric Hamiltonian satisfy commutation and anticommutation relations and form a superalgebra. The particular symmetry scheme obtained this way, i.e. supersymmetry can then be accounted for the degeneracies occurring in the spectra. The most frequently used model is $N = 2$ SUSYQM, in which the two supercharges and the supersymmetric Hamiltonian are represented in terms of 2×2 matrices [5, 8].

The construction leading to isospectral Hamiltonians can be generalized in several ways. The factorization energy, for example, can be chosen to be different from zero: $H_1 = QR + \epsilon$ [9], or differential operators of higher order can be considered in the realization of the Q- and R-type operators [3, 10]. Here we focus on the factorization of the three-dimensional Schrödinger equation, rather than on that of the one-dimensional one. We construct the operators Q and R in a rather general form and systematically explore the possibilities by specifying their form according to various requirements (hermiticity, symmetries and invariances with respect to various operations). This construction turns out to be especially suited to describing quantum mechanical potential problems also depending on spin degrees of freedom.

In section 2 we discuss the factorization of the Schrödinger equation in terms of a pair of matrix valued linear differential operators and specify the formulae for central potentials. In section 3 three examples are presented and the characteristic degeneracies of the spectra are discussed. The results are summarized in section 4, where our approach is also placed in a wider context. A brief discussion of some specific non-central problems is given in the appendix.

2. The basic formalism

Consider the factorization of the Hamiltonians (2) in terms of the operators Q and R defined as

$$Q = \sigma \cdot (\boldsymbol{p} + \boldsymbol{a}(\boldsymbol{r})) + C(r) \qquad R = \sigma \cdot (\boldsymbol{p} + \boldsymbol{b}(\boldsymbol{r})) + D(r) \tag{6}$$

with units $\hbar = 2m = 1$. Now assume that C and D are functions of $r = |\boldsymbol{r}|$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ have the form

$$\boldsymbol{a}(\boldsymbol{r}) = f(r)\boldsymbol{r} \qquad \boldsymbol{b}(\boldsymbol{r}) = g(r)\boldsymbol{r}. \tag{7}$$

This choice naturally leads to potential problems with spherical symmetry. Substituting (7) into (6) one finds that

$$H_1 = \boldsymbol{p}^2 + (g+f)\boldsymbol{r}\cdot\boldsymbol{p} + \mathrm{i}(f-g)\sigma\cdot\boldsymbol{L} - \mathrm{i}g'r - 3\mathrm{i}g + gfr^2 + CD$$
$$+(C+D)\sigma\cdot\boldsymbol{p} + \left(fD + gC - \mathrm{i}\frac{1}{r}D'\right)\sigma\cdot\boldsymbol{r}. \tag{8}$$

Note that the last two terms of H_1 in (8) have pseudoscalar character. Also note the appearance of terms usually considered are of relativistic origin. The corresponding formula for H_2 readily follows from (8) by the $f \leftrightarrow g$ and $C \leftrightarrow D$ replacements.

Table 1 summarizes the conditions under which some of the terms vanish, and also lists the consequences of certain prescribed properties of Q and R. These latter ones include conditions which guarantee the Hermiticity of H_1 and H_2. (Besides the standard $R^\dagger = Q$ choice we also list the more special $R^\dagger = Q = R$ case, which recovers the Pauli Hamiltonian that can be formulated in terms of $N = 1$ SUSYQM [11] using a single self-adjoint supercharge operator.) Table 1 also lists the condition for time-reversal invariance requiring that the terms including $\sigma\cdot\boldsymbol{p}$ and $\sigma\cdot\boldsymbol{r}$ transform in the same way under time reversal. Conditions for the more general case of non-central potentials could also be formulated. A particular case of such problems is discussed briefly in the appendix.

In what follows we focus on problems where the resulting Hamiltonians are Hermitian, have spherical symmetry, and are free from pseudoscalar and explicitly linear derivative terms. (We note that non-Hermitian Hamiltonians are also considered in various areas [12], such as nuclear systems admitting optical (complex) potentials [13].) Simple calculations show that the above conditions are met if $C(r) = D(r) = 0$ and $g(r) = -f(r) = f^*(r)$ hold. (This latter condition restricts the $f(r)$ function to purely imaginary values.) The Hamiltonians obtained this way depend on the unspecified function $f(r)$ and describe two non-relativistic problems with spin–orbit interaction:

$$H_1 = \boldsymbol{p}^2 + 2\mathrm{i}f\sigma\cdot\boldsymbol{L} + \mathrm{i}f'r + 3\mathrm{i}f - f^2r^2 \tag{9}$$

$$H_2 = \boldsymbol{p}^2 - 2\mathrm{i}f\sigma\cdot\boldsymbol{L} - \mathrm{i}f'r - 3\mathrm{i}f - f^2r^2. \tag{10}$$

It is natural to express the wavefunctions in terms of orbital and spin states as

$$|nljm\rangle = \sum_{Ms}\langle lM\tfrac{1}{2}s|jm\rangle|nlM\rangle|\tfrac{1}{2}s\rangle \tag{11}$$

Table 1. Conditions guaranteeing certain properties of operators Q and R as defined by (6) and (7) and those of Hamiltonians $H_1 = QR$ and $H_2 = RQ$.

Prescription		Conditions	
Properties of Q and R	$Q^\dagger = Q$	$f^*(r) = f(r)$	$C^*(r) = C(r)$
	$R^\dagger = R$	$g^*(r) = g(r)$	$D^*(r) = D(r)$
	$R^\dagger = Q$	$f^*(r) = g(r)$	$C^*(r) = D(r)$
	$R = Q$	$f(r) = g(r)$	$C(r) = D(r)$
	Time reversal invariance	$f^*(r) = -f(r)$	$g^*(r) = -g(r)$
Properties of H_1 and H_2	no $\sigma\cdot\boldsymbol{p}$ term		$C(r) = -D(r)$
	no $\sigma\cdot\boldsymbol{r}$ term	$f(r)D(r) + g(r)C(r) - \mathrm{i}r^{-1}D'(r) = 0$	$C'(r) = D'(r)$
	no $\sigma\cdot\boldsymbol{p}$ and $\sigma\cdot\boldsymbol{r}$ term	either $g(r) = f(r)$ and	$C(r) = -D(r) = \text{const.} \neq 0$
		or	$C(r) = D(r) = 0$
	no $\boldsymbol{r}\cdot\boldsymbol{p}$ term	$g(r) = -f(r)$	

where the coordinate representation of the orbital states can be written as the product of a spherical harmonics and a radial wavefunction:

$$\langle \boldsymbol{r}|nlM\rangle = R_{nl}(r)Y_{lM}(\theta,\phi). \tag{12}$$

In what follows we assume that the radial wavefunctions of the bound states are normalized in the following way:

$$\int_0^\infty R_{n'l}(r)R_{nl}(r)\,\mathrm{d}r = \delta_{n'n}. \tag{13}$$

Straightforward tensor algebraic calculations reveal (see e.g. [14]) that the matrix elements of $Q = \sigma\cdot(\boldsymbol{p}+f(r)\boldsymbol{r})$ and $R = \sigma\cdot(\boldsymbol{p}-f(r)\boldsymbol{r})$ in the basis defined in (11–13) are given by

$$\langle n'(l+1,\tfrac{1}{2})jm|\sigma\cdot(\boldsymbol{p}\pm f(r)\boldsymbol{r})|n(l\tfrac{1}{2})jm\rangle = \int_0^\infty R_{n'l+1}(r)\left[-\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l+1}{r}\right)\pm rf(r)\right]R_{nl}(r)\,\mathrm{d}r \tag{14}$$

$$\langle n'(l-1,\tfrac{1}{2})jm|\sigma\cdot(\boldsymbol{p}\pm f(r)\boldsymbol{r})|n(l\tfrac{1}{2})jm\rangle = \int_0^\infty R_{n'l-1}(r)\left[-\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l}{r}\right)\pm rf(r)\right]R_{nl}(r)\,\mathrm{d}r. \tag{15}$$

Since Q and R are pseudoscalar operators, the matrix elements are diagonal in the quantum numbers j and m, but the parity of the bra and ket states must be different. This difference is also demonstrated by the change of the orbital angular momentum l by one unit. In the general case no selection rules can be given for the principal quantum numbers n and n'.

Since the formalism of the factorization method applies to any construction satisfying (3), irrespective of the realization of the operators, H_1 and H_2 defined by (2) have to be essentially isospectral, in the sense specified in section 1. This statement equally applies to analytically solvable problems and those admitting only numerical solutions. In the next section we present examples for which bound state solutions (or at least, part of them) can be given analytically.

3. Examples

3.1. The harmonic oscillator case

Substituting $f(r) = \mathrm{i}\omega/2$ in (9) and (10) one gets

$$H_1 = \boldsymbol{p}^2 - \omega\sigma\cdot\boldsymbol{L} - \frac{3}{2}\omega + \frac{\omega^2}{4}r^2 \tag{16}$$

$$H_2 = \boldsymbol{p}^2 + \omega\sigma\cdot\boldsymbol{L} + \frac{3}{2}\omega + \frac{\omega^2}{4}r^2. \tag{17}$$

These equations describe oscillators, which also experience spin–orbit interaction, the strength of which is correlated with the oscillator constant. Acting with Hamiltonians (16) and (17) on wavefunctions of the type (12) with $j = l+\frac{1}{2}$ and $j = l-\frac{1}{2}$, we get radial Schrödinger equations of harmonic oscillator type, from which the energy eigenvalues can immediately be determined:

$$E_{nl}^{(1+)} = 2\omega n \qquad E_{nl}^{(1-)} = \omega(2n+2l+1) \tag{18}$$

$$E_{nl}^{(2+)} = \omega(2n+2l+3) \qquad E_{nl}^{(2-)} = 2\omega(n+1). \tag{19}$$

Here the superscripts '+' and '−' stand for states with $j = l+\frac{1}{2}$ and $j = l-\frac{1}{2}$, respectively, while 1 and 2 specify states belonging to H_1 and H_2. Note that both the orbital angular momentum l and the radial quantum number n can take any non-negative integer value.

The same results can be obtained by calculating the matrix elements of H_1 and H_2 through those of R and Q using (14) and (15):

$$\langle n'(l+1,\tfrac{1}{2})jm|Q|n(l,\tfrac{1}{2})jm\rangle = \mathrm{i}\delta_{n',n}[\omega(2n+2l+3)]^{\frac{1}{2}} \tag{20}$$

$$\langle n'(l-1,\tfrac{1}{2})jm|Q|n(l,\tfrac{1}{2})jm\rangle = -\mathrm{i}\delta_{n',n+1}[2\omega(n+1)]^{\frac{1}{2}} \tag{21}$$

$$\langle n'(l+1,\tfrac{1}{2})jm|R|n(l,\tfrac{1}{2})jm\rangle = \mathrm{i}\delta_{n',n-1}(2\omega n)^{\frac{1}{2}} \tag{22}$$

$$\langle n'(l-1,\tfrac{1}{2})jm|R|n(l,\tfrac{1}{2})jm\rangle = -\mathrm{i}\delta_{n',n}[\omega(2n+2l+1)]^{\frac{1}{2}}. \tag{23}$$

These equations also display the selection rules applicable to the radial quantum number n in the harmonic oscillator case, furthermore, they confirm that $Q = R^{\dagger}$. It is also clear that Q acts as the inverse of R (and vice versa), up to a phase factor $\pm\mathrm{i}$. Note that the right-hand side of (22) is zero if R acts on the state $|n = 0(l,\frac{1}{2})jm\rangle$ if $j = l + \frac{1}{2}$, irrespective of l. Since this is the lowest possible energy eigenvalue of H_1 (see (18)), we can conclude that the ground state of this problem is infinitely degenerate and also that the corresponding energy levels are missing from the spectrum of H_2.

The eigenstates of H_1 and H_2 belong to two different groups. The first group contains the states with $E^{(1+)}_{n,l} = E^{(2-)}_{n-1,l+1} = 2\omega n$ ($n = 1, 2, \ldots$) and the ground state of H_1 with $E^{(1+)}_{n=0,l} = 0$. These states are infinitely degenerate, because there is a state with this energy for any l. The states belonging to the second group are located at other energies: $E^{(1-)}_{n,l} = E^{(2+)}_{n,l-1} = \omega(2n + 2l + 1)$. These levels have finite degeneracy, because degeneracy requires $n + l =$ const., which can be obtained only in a finite number of ways. It is easy to see that the degree of degeneracy is $k(k + 1)$ for the states with $n + l = k$. The structure of the two spectra is displayed in figure 1.

Similar results have been obtained by Quesne [15], who studied the Dirac oscillator [16]. In fact, the squared Dirac equation, in this case results in equations like (16) and (17). The degeneracy of the energy eigenvalues originates in both approaches from the correlated strength of the oscillator parameter and the spin–orbit coupling constant. Balantekin has also analyzed this problem in terms of SUSYQM [17] and interpreted its 'accidental' degeneracies as arising from an $SU(2) \times Osp(2/2)$ symmetry group.

We also note that the selection rules concerning the orbital quantum numbers n and l shown in (20–23) can also be interpreted in a straightforward way noting that $\boldsymbol{p} \pm \mathrm{i}\frac{\omega}{2}\boldsymbol{r}$ are the usual ladder operators of the harmonic oscillator problem. In particular, they connect states with $N' = N \pm 1$, where $N = 2n + l$. It is clear then, that $N' = N + 1$, for example, can be obtained from the combinations $n' = n$, $l' = l + 1$ and $n' = n + 1$, $l' = l - 1$.

3.2. The Coulomb case

The $f = \mathrm{i}cr^{-1}$ choice results in Hamiltonians in which the spin–orbit interaction appears in a Coulomb-like term, rather than in a constant one:

$$H_1 = \boldsymbol{p}^2 + c^2 - 2\frac{c}{r}(\sigma \cdot \boldsymbol{L} + 1) \tag{24}$$

$$H_2 = \boldsymbol{p}^2 + c^2 + 2\frac{c}{r}(\sigma \cdot \boldsymbol{L} + 1). \tag{25}$$

Evidently, bound states can only appear when the coefficient of the r^{-1} type term is negative. Without the loss of generality we can assume that $c > 0$ holds: a $c \to -c$ transformation merely interchanges the role of H_1 and H_2. Then the sign of the Coulomb term is determined by $\langle \sigma \cdot \boldsymbol{L} + 1\rangle$, which is $l + 1$ for $j = l + \frac{1}{2}$ and $-l$ for $j = l - \frac{1}{2}$. The resulting spectra are then

$$E^{(1+)}_{nl} = c^2\left(1 - \frac{(l+1)^2}{(n+l+1)^2}\right) \qquad E^{(1-)}: \text{no bound states} \tag{26}$$

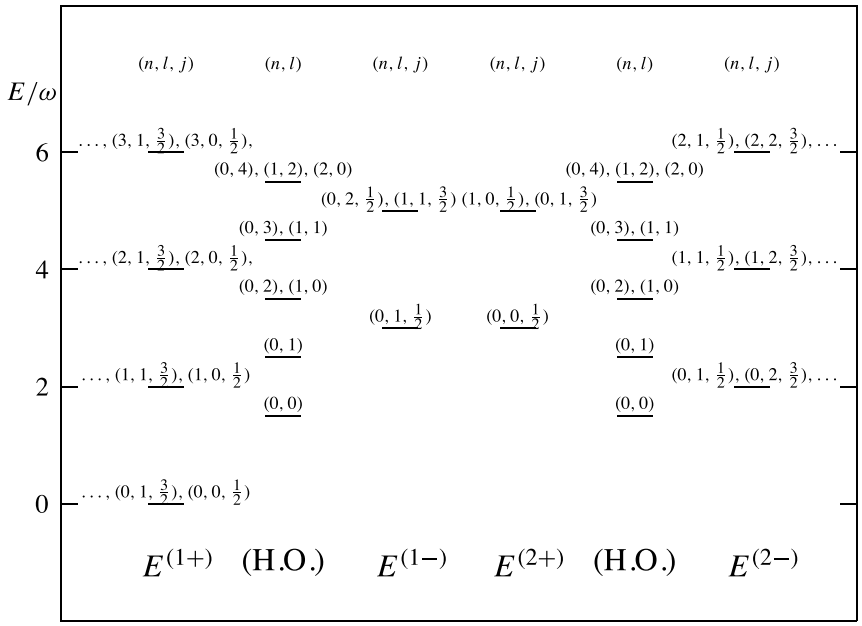

Figure 1. The energy spectrum of the oscillator problems discussed in section 3.1. The first few energy levels of H_1 and H_2 (denoted with quantum numbers (n, l, j) are depicted on the left- and the right-hand sides of the figure, respectively. The usual harmonic oscillator states (denoted with (n, l)) are also displayed, in order to elucidate the origin of the states obtained by spin–orbit coupling. The infinitely degenerate levels of H_1 and H_2 are on the two sides of the figure, while those with finite degeneracy are in the middle section.

$$E^{(2+)}: \text{no bound states} \qquad E_{nl}^{(2-)} = c^2\left(1 - \frac{l^2}{(n+l+1)^2}\right) \tag{27}$$

where the meaning of the superscripts is the same as in (18) and (19).

The matrix elements of R and Q are

$$\left\langle n'\left(l-1, \frac{1}{2}\right) jm|Q|n\left(l, \frac{1}{2}\right) jm\right\rangle = -\mathrm{i}\delta_{n',n+1}c\frac{[(n+1)(n+2l+1)]^{\frac{1}{2}}}{n+l+1} \tag{28}$$

$$\left\langle n'\left(l+1, \frac{1}{2}\right) jm|R|n\left(l, \frac{1}{2}\right) jm\right\rangle = \mathrm{i}\delta_{n',n-1}c\frac{[n(n+2l+2)]^{\frac{1}{2}}}{n+l+1}. \tag{29}$$

The energy eigenvalues of $H_1 = QR$ and $H_2 = RQ$ also readily follow from these matrix elements.

Similarly to the conventional Coulomb problem, the energy levels tend to a well defined value in the $n \to \infty$ limit. This value is not zero, rather it is c^2, due to the different choice of the energy scale. $E = 0$ now corresponds to the ground state of H_1 for the states with $j = l + \frac{1}{2}$, as can be seen from (26) with $n = 0$. This applies to any value of the orbital angular momentum l, so we again have an infinitely degenerate ground state for H_1. The corresponding energy levels are missing from the spectrum of H_2 again, as can be seen from (27).

The first few energy levels of H_1 and H_2 are plotted in figure 2. The energy levels again exhibit a complex degeneracy pattern: $E_{nl}^{(1+)}$ is the same whenever the ratio $(l+1)/(n+l+1)$ has the same value, which can be realized in an infinite variety of ways. (A special case of this is the degeneracy of the ground state with $n = 0$.) Similarly, $E_{nl}^{(2-)}$ has the same value if $l/(n+l+1)$ is fixed.

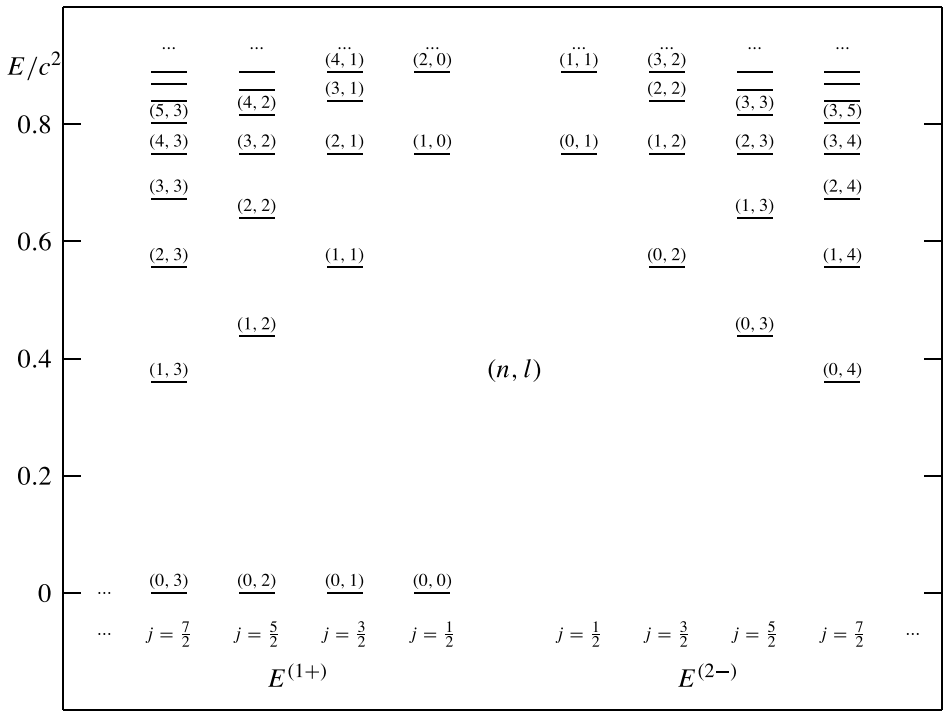

Figure 2. The bound-state energy spectrum of the Coulomb-like problems discussed in section 3.2. The eigenstates of H_1 with $j = l + \frac{1}{2}$ are displayed in the left-hand side of the figure, while those of H_2 with $j = l - \frac{1}{2}$ on the right-hand side. States with the same value of j appear in the same column in both cases. Only states with $E/c^2 \leqslant \frac{8}{9}$ and $j \leqslant \frac{7}{2}$ are shown.

3.3. Some quasi-exactly solvable problems

Besides the harmonic oscillator and Coulomb problems, quasi-exactly solvable (QES) potentials (for a review see [18]) offer further examples for which the solutions can be obtained (at least partly) for any value of l in principle. This is an essential requirement for the application of the formalism outlined in section 2, because the Q and R operators change the value of the orbital angular momentum with one unit. (See (14) and (15).) The quasi-exactly solvable models have infinite number of energy levels in general, but exact solutions can be obtained only for a finite set of them. Furthermore, this is possible only if the potential parameters satisfy certain conditions.

The most well known QES potential is the sextic oscillator, for which the radial Schrödinger equation is

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \left(\frac{l(l+1)}{r^2} + Ar^2 + Br^4 + Cr^6\right)\psi = E\psi. \tag{30}$$

The condition of quasi-exact solvability is

$$C = a^2 \qquad B = 2ab \qquad A = b^2 - 2a(2m + l + \tfrac{5}{2}). \tag{31}$$

Here $a > 0$ and m is a non-negative integer. The solutions ψ can then be written as

$$\psi(r) \simeq r^{l+1}\exp\left(-\frac{a^2}{4}r^4 - \frac{b}{2}r^2\right)\Phi_m(r^2) \tag{32}$$

where $\Phi_m(r^2)$ is an mth order polynomial of r^2 [19].

Hamiltonians of the type appearing in (30) can be obtained from (9) and (10) by substituting $f = \mathrm{i}\frac{\omega}{2} + \mathrm{i}\beta r^2$ in them:

$$H_1^{(+)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \left(\frac{\omega^2}{4} - \beta(2l+5)\right) r^2 + \beta\omega r^4 + \beta^2 r^6 - \omega\left(l + \frac{3}{2}\right) \tag{33}$$

$$H_1^{(-)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \left(\frac{\omega^2}{4} + \beta(2l-3)\right) r^2 + \beta\omega r^4 + \beta^2 r^6 + \omega\left(l - \frac{1}{2}\right) \tag{34}$$

$$H_2^{(+)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \left(\frac{\omega^2}{4} + \beta(2l+5)\right) r^2 + \beta\omega r^4 + \beta^2 r^6 + \omega\left(l + \frac{3}{2}\right) \tag{35}$$

$$H_2^{(-)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} + \left(\frac{\omega^2}{4} - \beta(2l-3)\right) r^2 + \beta\omega r^4 + \beta^2 r^6 - \omega\left(l - \frac{1}{2}\right). \tag{36}$$

The substitution $\beta = b$, $\omega = 2a$ brings these equations into a form similar to (30), with the exception of an energy shift. However, we find that the condition for quasi-exact solvability (as in (31)) is fulfilled only by (33), and only for $m = 0$. Thus, these solutions have the form of (32) with $m = 0$, i.e. the polynomial $\Phi(r^2)$ is reduced to a constant. This means that these nodeless functions are the ground state solutions of $H_1^{(+)}$, and we also find that the corresponding energy eigenvalues are zero. Furthermore, these observations apply for states with any l, therefore the situation is very similar to that described in sections 3.1 and 3.2: H_1 has infinitely degenerate ground state with $E = 0$ for $j = l + \frac{1}{2}$, and this energy level is missing from the spectrum of the partner Hamiltonian $H_2^{(-)}$, i.e. H_2 in the $j = l - \frac{1}{2}$ case. The remaining solutions cannot be described in terms of this approach. This applies also to the solutions of $H_1^{(-)}$ and $H_2^{(+)}$. (Note that the role of H_1 and H_2 is interchanged if the $\beta = -b$, $\omega = -2a$ choice is made, which simply replaces $f(r)$ with $-f(r)$. Also note that $\beta = 0$ reduces the problem to that discussed in section 3.1.)

The Hamiltonian $H_1^{(+)}$ in (33) is similar to the second interpretation of the Hamiltonian described in [19]: the linear dependence on l there is attributed to spin–orbit coupling term. Then one component (with $j = l + \frac{1}{2}$) of the multi-channel problem can be described in the QES framework. It is remarkable that this QES problem has been derived in two completely different ways. The author of [19] proposed the incorporation of the spin–orbit term in the sextic oscillator problem to account for the linear l dependence of the r^2 term, while in our approach the spin–orbit term appeared by construction in the sextic oscillator problem, when an appropriate choice of $f(r)$ was made in (9).

Similar considerations can be made for other QES problems as well. The $f = \mathrm{i}s + \mathrm{i}tr^{-1}$ choice leads to Hamiltonians containing a (shifted) oscillator potential and $\sigma \cdot \boldsymbol{L}$ dependent Coulomb-like and constant terms. (The $t = 0$ and the $s = 0$ choices, of course, reduce this problem to those discussed in sections 3.1 and 3.2, respectively.) Here, again we find that only the zero-energy, infinitely degenerate ground state of H_1 can be obtained exactly in this framework, and only for $j = l + \frac{1}{2}$. It is remarkable that a situation rather similar to this has been found by analysing a QES problem, the sextic oscillator in relativistic quantum mechanical framework, as a departure from the Dirac oscillator [20]: the positive-energy spectrum had an infinitely degenerate level at $E = m$ for $j = l + \frac{1}{2}$.

4. Concluding remarks

We presented a generalization of the factorization method of non-relativistic Hamiltonians in terms of matrix valued linear differential operators. This method naturally introduces spin degrees of freedom in the Hamiltonians, which exhibit the same kind of relations as in the

conventional factorization method: each bound state of H_1 has equivalents in the spectrum of H_2, except for the zero-energy ground state. A characteristic feature of this approach is that a spin–orbit term is automatically introduced in the Hamiltonians. We note here that the spin–orbit interaction also appears naturally in Hamiltonians constructed using symmetry-based algebraic techniques [21].

The shift operators connecting the states of H_1 and H_2 are pseudoscalar operators, therefore they leave the full angular momentum j invariant, but change the parity of the state and modify the orbital angular momentum l with one unit. This change of l is a characteristic feature of the conventional factorization method and SUSYQM too: the partner potentials of radial problems there have centrifugal terms formally containing l and $l+1$. However, it has been noted [22] that this change can only be formal, because the value of l is fixed when the radial Schrödinger equation is written down, therefore the centrifugal term should remain unchanged during the whole procedure and the additional l-dependent singular terms are to be interpreted dynamically [23]. In contrast with this scenario, in our procedure the complete Schrödinger equation is factorized, including the full kinetic energy term, therefore the change of l is not formal.

Besides all this, the similarity with the conventional factorization and SUSYQM techniques is striking: the linear differential operators remaining in (14) and (15) after the integration of the angular variables are practically the same as the A and $A^\dagger$ operators of SUSYQM. (The imaginary factors i and $-$i could be eliminated by a trivial redefinition of Q and R.) In fact, it can be shown that the radial Schrödinger equation obtained from H_1 in (9) with $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ can be obtained by formally substituting $W(r) = -\mathrm{i}rf(r) - (l+1)r^{-1}$ and $W(r) = -\mathrm{i}rf(r) + lr^{-1}$ in the 'bosonic' Hamiltonian $H_- = A^\dagger A = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + W^2(r) - W'(r)$ of SUSYQM. Similar connection can be established between H_2 in (10) and the 'fermionic' Hamiltonian, the SUSYQM partner of H_-.

The whole procedure outlined in section 2 could, of course, be repeated in a SUSYQM formalism. In the simplest case Q and R could be incorporated in the two supercharges as off-diagonal elements of 2×2 matrices and H_1 and H_2 would become the bosonic and fermionic component of the supersymmetric Hamiltonian. The specific case considered in detail in sections 2 and 3 (with $C = D = 0$ and $g = -f$) would then become the one discussed by Beckers *et al* [24], who studied the conditions under which three-dimensional supersymmetric Schrödinger equations with central potentials lead to supersymmetric one-dimensional radial equations. Similar constructions have also been considered in [25] in connection with the discussion of SUSYQM in two and three dimensions.

The potential problems considered in the present framework need not be exactly solvable. All conclusions concerning isospectrality of the partner Hamiltonians hold for problems that can only be solved numerically. However, for the sake of simplicity we chose exactly solvable cases as illustrative examples: the harmonic oscillator, the Coulomb and some quasi-exactly solvable problems. This is because the resulting Schrödinger-like equations can be solved for any value of l in these cases, which is important to demonstrate the merits of our approach. (Most of the exactly solvable potentials are, in fact, solvable only for $l = 0$.) Although a complete solution could only be obtained for the oscillator and the Coulomb cases, there seem to be certain common characteristic features of all these problems. Perhaps the most important one is that the physical systems have an infinitely degenerate zero-energy ground state, which appears for $j = l + \frac{1}{2}$ in one of the partner Hamiltonians. In the terminology of SUSYQM this could heuristically be stated as having an infinite Witten index [26].

The fact that the spin–orbit term is related to the other potential terms through $f(r)$ allows interesting combinations in the Hamiltonians. One can use this freedom to generate a pair of

Hamiltonians in which one, say H_1 has only kinetic and spin–orbit terms. Such Hamiltonians may have relevance in discussing pseudospin symmetry of nuclei, for example [27]. The particular f that reduces H_1 to $H_1 = \boldsymbol{p}^2 + 2\mathrm{i}f\sigma\cdot\boldsymbol{L}$ is $f(r) = \mathrm{i}ar^{-2}(r+a)^{-1}$, with $a > 0$ to avoid singularities. H_2 then has a spin–orbit term with opposite sign and also some extra potential terms. Based on the construction, the two Hamiltonians should then be isospectral, except for the ground state. However, simple analytical calculations show that the potentials appearing in all four radial Schrödinger equations are repulsive, so no bound states are possible. The analogous situation can occur in SUSYQM too: choosing a particular form of the superpotential $W(r)$ may result in a potential with a known solution, at least at $E = 0$, but it is not guaranteed that the solutions of this potential and its SUSYQM partner are normalizable and bound physical states appear in the spectra. In fact, this is what happens if we define $W(r)$ in terms of $f(r)$ as we have discussed earlier in this section.

The formalism developed here is applicable to non-central problems as well. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in (6) can be defined in more general ways as in (7). One possibility (described in the appendix) is adding a $(\boldsymbol{r}\times\boldsymbol{B})$-type to them, in which case the Hamiltonians are supplemented with terms describing interactions in the presence of an external magnetic field collinear with vector $\boldsymbol{B}$.

Appendix

Here we consider a special generalization of the factorized Hamiltonians by including a further vector term in $\boldsymbol{a}$ and $\boldsymbol{b}$ of (7):

$$\boldsymbol{a}(\boldsymbol{r}) = f(r)\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{A} \qquad \boldsymbol{b}(\boldsymbol{r}) = g(r)\boldsymbol{r} + \boldsymbol{r}\times\boldsymbol{B}. \tag{37}$$

We assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are axial vectors, so that the vectorial product has polar vector character. As a further simplification we also suppose that the two new vectors are constant in space and have parallel direction: $\boldsymbol{A} = A\boldsymbol{n}$, $\boldsymbol{B} = B\boldsymbol{n}$ with A and B constants. The conditions securing the specific relations between Q and R listed in table 1, i.e. $Q^\dagger = Q$, $R^\dagger = R$, $R^\dagger = Q$ and $R = Q$ then have to be supplemented with $A^* = A$, $B^* = B$, $A^* = B$ and $A = B$, respectively. With $A = B = real$ all these relations are satisfied simultaneously.

With the restrictions described here, the following new terms have to be added to H_1 in (8):

$$\begin{aligned}\tilde{H}_1 = {} & -2\boldsymbol{B}\cdot(\boldsymbol{L}+\sigma) + r^2B^2 - (\boldsymbol{r}\cdot\boldsymbol{B})^2 + \mathrm{i}(f-g)(\sigma\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{B}) \\ & +\mathrm{i}(g-f)r^2\sigma\cdot\boldsymbol{B} + (C+D)\sigma\cdot(\boldsymbol{r}\times\boldsymbol{B}).\end{aligned} \tag{38}$$

With the exception of the last term, (38) contributes to the scalar part of H_1 displayed in (8). (The corresponding expressions for H_2 are obtained by interchanging $f(r)$ with $g(r)$ and $C(r)$ with $D(r)$.) The first term here can be interpreted as a magnetic dipole-type interaction with a notable spin giromagnetic factor of two. If the vector field $\boldsymbol{B}$ is allowed to be dependent on $\boldsymbol{r}$, some extra derivative terms also appear in $\tilde{H}_1$.

Here we mention two specific cases. The first one is the choice made in section 2 to get rid of the $\boldsymbol{r}\cdot\boldsymbol{p}$ and the pseudoscalar terms: $C(r) = D(r) = 0$, $g(r) = -f(r)$. Then, after some algebra

$$\tilde{H}_1 = -2\boldsymbol{B}\cdot(\boldsymbol{L}+\sigma) + (\boldsymbol{r}\times(\boldsymbol{B}-\mathrm{i}f\sigma))^2 + 2f^2r^2 \tag{39}$$

follows. As another possibility $C(r) = D(r) = 0$ and $g(r) = f(r)$ can be chosen, in which case the pseudoscalar terms vanish, together with the spin–orbit term in (8), while the $\boldsymbol{r}\cdot\boldsymbol{p}$ term survives:

$$\tilde{H}_1 = -2\boldsymbol{B}\cdot(\boldsymbol{L}+\sigma) + r^2B^2 - (\boldsymbol{r}\cdot\boldsymbol{B})^2. \tag{40}$$

Note that R and Q become identical in this case, just as $H_1 = R^2$ and $H_2 = Q^2$, so the isospectrality of the two Hamiltonians becomes trivial.

Although the new terms break the spherical symmetry of the problem, the basis defined in (11) and (12) can still be used to determine the matrix elements of R, Q, H_1 and H_2 by relatively straightforward tensor algebraic calculations.

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References

[1] Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 9
Schrödinger E 1941 *Proc. R. Irish Acad.* A **46** 83
Schrödinger E 1941 *Proc. R. Irish Acad.* A **47** 53
Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[2] Carrol R W 1979 *Transmutation and Operator Differential Equations (North Holland Mathematics Studies vol 37)* (Amsterdam: North Holland)
Bagrov V G and Samsonov B F 1996 *Phys. Lett.* A **210** 60
Cariñena J F, Marmo G, Perelomov A M and Rañada M F 1998 *Int. J. Mod. Phys.* A **13** 4913
[3] Fernandez C D J 1997 *Int. J. Mod. Phys.* A **12** 171
Fernandez C D J, Glasser M L and Nieto L M 1998 *Phys. Lett.* A **240** 15
Andrianov A A, Cannata F, Ioffe M V and Nishnianidze D N 1997 *J. Phys. A: Math. Gen.* **30** 5037
[4] Bohm A, Ne'eman Y and Barut A O ed 1988 *Dynamical Groups and Spectrum-Generating Algebras* (Singapore: World Scientific)
[5] Lahiri A, Roy P K and Bagchi B 1990 *Int. J. Mod. Phys.* A **5** 1383
Lévai G 1994 *Quantum Inversion Theory and Applications (Lecture Notes in Physics vol 427)* ed H V von Geramb (Berlin: Springer) p 107
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[6] Nieto M M 1984 *Phys. Lett.* B **145** 208
Andrianov A A, Borisov N B and Ioffe M V 1984 *Phys. Lett.* A **105** 19
[7] Lévai G 1994 *J. Phys. A: Math. Gen.* **27** 3809
Gangopadhyaya A, Mallow J and Sukhatme U P 1998 *Proc. Workshop on Supersymmetric Quantum Mechanics and Integrable Models* ed H Aratyn *et al* (Berlin: Springer)
Balantekin A B 1998 *Phys. Rev.* A **57** 4188
[8] Witten E 1981 *Nucl. Phys.* B **188** 513
[9] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
[10] Andrianov A A, Ioffe M V and Spiridonov V 1993 *Phys. Lett.* A **174** 273
Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 *Int. J. Mod. Phys.* A **10** 2683
[11] de Combrugghe M and Rittenberg V 1983 *Ann. Phys., NY* **151** 99
[12] Cannata F, Junker G and Trost J 1998 *Phys. Lett.* A **246** 219
Andrianov A A, Cannata F, Dedonder J-P and Ioffe M V 1998 *Preprint* quant-phys/9806019 *Int. J. Mod. Phys.* to be published
[13] Baye D, Lévai G and Sparenberg J-M 1996 *Nucl. Phys.* A **599** 435
[14] Edmonds A R 1960 *Angular Momentum in Quantum Mechanics* (Princeton, NJ: Princeton University Press)
[15] Quesne C 1991 *Int. J. Mod. Phys.* A **6** 1567
[16] Cook P A 1971 *Lett. Nuovo Cimento* **1** 419
Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[17] Balantekin A B 1985 *Ann. Phys., NY* **164** 277
[18] Ushveridze A G 1994 *Quasi-Exactly Solvable Problems in Quantum Mechanics* (Bristol: IOP)
[19] Ushveridze A G 1998 *Preprint* hep-th/9810045
[20] Lévai G and del Sol Mesa A 1996 *J. Phys. A: Math. Gen.* **29** 2827
[21] Lévay P 1995 *J. Phys. A: Math. Gen.* **28** 5919
[22] Baye D 1987 *J. Phys. A: Math. Gen.* **20** 5529
[23] Amado R D, Cannata F and Dedonder J-P 1990 *Int. J. Mod. Phys.* A **5** 3401

[24] Beckers J, Debergh N and Nitbashirakandi L 1995 *J. Math. Phys.* **36** 641
[25] Das A, Okubo S and Pernice S A 1997 *Mod. Phys. Lett.* A **12** 581
[26] Witten E 1982 *Nucl. Phys.* B **202** 253
[27] Ginocchio J N and Leviatan A 1998 *Phys. Lett.* B **425** 1